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A Formal Inverse to the Cayley–Hamilton Theorem

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We prove that a Q-algebra R with formal trace can be realized as $n \times n$ matrices if and only if it satisfies the Cayley–Hamilton identity of degree n. © 1987 Academic Press, Inc.

0. INTRODUCTION

0.1. The purpose of this paper is to analyze the following question: Given a ring R and an integer n under which conditions can we embed R as a subring of the ring $M_n(A)$ of $n \times n$ matrices over a commutative ring A?

The theory of polynomial identities sheds some light on this question: a necessary condition for the existence of such an embedding is that R satisfies all polynomial identities satisfied by $n \times n$ matrices (over Z).

0.2. The previous condition is in general not sufficient (cf. [6]) and a complete answer to the embedding problem is not known; nevertheless there are some remarkable theorems known when one adds some further restrictions to R. We want to recall two basic theorems, assuming that R satisfies the identities of $n \times n$ matrices we have:

THEOREM 1 (Posner). *If R is a prime ring then R is an order in a central simple algebra of degree $\leqslant n$.*

THEOREM 2 (Artin). *If no factor of R satisfies the polynomial identities of $n-1 \times n-1$ matrices, R is an Azumaya algebra over its center of constant rank n^2.*

In each of the previous instances we can use the structure theorem for precise answers to the embedding problem.

0.3. In this paper we take a more formal point of view. We consider the category of algebras with trace and will explain in Section 2 what a Q-algebra with trace satisfying the Cayley–Hamilton identity of degree n

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means. (This setting is taken from the theory of trace identities.) We then prove the main result:

THEOREM. *A Q-algebra with trace can be embedded in $M_n(A)$, A commutative, if and only if it satisfies the Cayley–Hamilton identity of degree n.*

The restriction to characteristic zero is almost certainly essential and it will appear in this work as connected with the theory of linearly reductive algebraic groups.

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1. UNIVERSAL MAPS

1.1. It is useful to formulate the embedding problem in a more categorical form. We recall some basic facts about the point of view of generic matrices (for more details, cf. [3]).

Let us work in the category of algebras over a commutative ring F. Given an algebra R there is a universal map

$$j\colon \quad R \to M_n(A),$$

A commutative. The universal property is the following: Given a map $f\colon \ R \to M_n(B)$, B commutative, there exists a unique map $\bar{f}\colon \ A \to B$ making the following diagram commutative:

$$\begin{tikzcd} R \arrow[rr, "j"] \arrow[dr, "f"'] & & M_n(A) \arrow[dl, "M_n(\bar{f})"] \\ & M_n(B) & \end{tikzcd}$$

The basic although trivial remark is: R can be embedded in $n \times n$ matrices over a commutative ring if and only if j is injective.

1.2. There is an extra structure associated with j. Let $G = GL(n, F)$ be the group of invertible $n \times n$ matrices over F. For every commutative algebra B, G acts as a group of inner automorphisms on $M_n(B)$. In particular consider the map $\pi_g\colon M_n(A) \to M_n(A)$ given by conjugation by g. From the universal property we have a map $\varphi_g\colon A \to A$ such that

$$\pi_g \cdot j = M_n(\varphi_g) \cdot j.$$

Clearly if $g, h \in G$ we have

$$\pi_{gh} \cdot j = \pi_g \cdot \pi_h \cdot j = \pi_g \cdot M_n(\varphi_h) \cdot j = M_n(\varphi_h) \cdot \pi_g \cdot j$$
$$= M_n(\varphi_h)\, M_m(\varphi_g) \cdot j = M_n(\varphi_h \cdot \varphi_g)\, j.$$

So $\varphi_{gh} = \varphi_h \cdot \varphi_g$ and setting $\rho_g = \pi_g \cdot M_n(\varphi_g)^{-1}$ we have

PROPOSITION. *G acts, via the representation ρ_g, as a group of automorphisms of $M_n(A)$ and $j(R)$ is contained in the ring of invariants.*

1.3. A particularly important case of the previous construction occurs when $R = F\{x_i\}_{i \in I}$ is a free algebra. In this case A is a polynomial ring in variables $\xi_{j,k}^{(i)}$, $i \in I$, $j, j = 1,..., n$, $j(x_i)$ is the "generic matrix" with j, k entry the variable $\xi_{j,k}^{(i)}$.

The group action can also be given a different interpretation. We think of $M_n(A)$ as the ring of polynomial maps f: $M_n(B)^I \to M_n(B)$ defined over F (B any commutative ring) and the group action is given in the natural form $f^g((x_i)) = gf((g^{-1}x_i f)) \cdot g^{-1}$ so that $f = f^g$ means that f is an equivariant map.

We will return presently to the analysis of the ring of equivariant maps.

2. ALGEBRAS WITH TRACE

2.1. The notion of an algebra with trace is a typical example of ideas of universal algebra. We sketch some basic facts here and refer to [5] for an exposition which includes some historical remarks and bibliography (the reader should note the unfortunate transposition between pages in [5]).

The necessity for introducing this notion comes from the attempt to relate the formal identities of matrices to the Cayley–Hamilton identity.

2.2. If $X \in M_n(A)$ is a matrix over a commutative ring A its characteristic polynomial is defined by the formula $\chi_X(t) = \det(X - t)$. The Cayley–Hamilton theorem gives the basic relation

$$\chi_X(X) = 0.$$

In characteristic 0 the polynomial $\chi_X(t)$ can be computed using the elements $\operatorname{tr}(X^i)$, $i = 1, 2,..., n$; for instance when $n = 2$ we have

$$\chi_X(t) = t^2 - \operatorname{tr}(x)\, t + \tfrac{1}{2}(\operatorname{tr}(x)^2 - \operatorname{tr}(x^2)).$$

The method is based on the formal recursive algorithm expressing elementary symmetric functions in terms of Newton functions, usually expressed by the formulae:

$$f(t)=\prod_{i=1}^{n}(t-t_i),$$

$$\frac{f'(t)}{f(t)}=\frac{d\log f(t)}{dt}=\sum_{i=1}^{n}\frac{1}{t-t_i}=\sum_{k=0}^{\infty}\frac{1}{t^{k+1}}\left(\sum_{i=1}^{n}t_i^k\right).$$

The formula can be used alternatively to compute the elements $\sum t_i^k$ in terms of the coefficients of $f(t)$ (in all characteristics) or in characteristic 0 to express $f(t)$ in terms of the elements $\sum t_i^k$. For the characteristic polynomial of X one only needs to substitute $\sum t_i^k$ with $\operatorname{tr}(X^k)$.

2.3. This formal approach lends itself to the methods of universal algebra so we set a definition:

DEFINITION. An F-algebra with trace is an algebra R equipped with an F linear map $\operatorname{tr}: R \to R$ satisfying the following axioms for $a, b \in R$:

(1) $\operatorname{tr}(a)\,b = b\operatorname{tr}(a)$,

(2) $\operatorname{tr}(ab)=\operatorname{tr}(ba)$,

(3) $\operatorname{tr}(\operatorname{tr}(a)\,b)=\operatorname{tr}(a)\operatorname{tr}(b)$.

2.4. Given an algebra with trace R (over Q), an integer n and an element $r \in R$ we can define a formal n-characteristic polynomial $\chi_r^{(n)}(t) = \prod_{i=1}^{n}(t-t_i)$ setting $\sum_{i=1}^{n} t_i^k = \operatorname{tr}(r^k)$.

In general this polynomial has no particular significance and it is introduced here as a test to see if R and tr are related to $n \times n$ matrices.

2.5. We consider the algebras with trace as a category. Maps between such algebras are trace preserving. The ring $M_n(A)$ of matrices over a commutative ring is considered with its natural trace. Thus it is natural in this context to formulate the embedding problem in $M_n(A)$. With respect to the general question from which we started we now have a further constraint, i.e., the embedding must preserve the trace. This is where the Cayley–Hamilton theorem will come into play.

2.6. DEFINITION. We say that an algebra R with trace satisfies the nth Cayley–Hamilton identity if for every $r \in R$ we have $\chi_r^{(n)}(r)=0$.

We can now state our main theorem in a more precise form.

THEOREM 2.6. *If R is a Q-algebra with trace satisfying the nth Cayley–Hamilton identity and $j: R \to M_n(A)$ is the universal map we have $j: \quad R \to M_n(A)^G \qquad$ is an isomorphism.*

Thus we not only have a solution to the embedding problem but in a way we also have an invariantly defined embedding.

2.7. Before proving the theorem we need to recall more of the theory of trace identities. Let us assume $F = Q$ and return to the construction of generic matrices discussed in 1.3. Let $B = M_n(A)$ denote the ring of polynomial maps $M_n(Q)^I \to M_n(Q)$ and $T = B^G$ the equivariant maps. We indicate by x_i the ith coordinate map, clearly $x_i \in T$.

The main facts about T are the first and second fundamental theorems of matrix invariants:

THEOREM 1. *T is generated as an algebra by the elements x_i and* $\operatorname{tr}(x_{i_1},..., x_{i_k})$, $k \leqslant n^2$.

THEOREM 2. *T is the free algebra with trace modulo the* nth *Cayley–Hamilton trace identity.*

These theorems can also be expressed as

THEOREM 3. *T is the free algebra in the generators x_i in the category of algebras with trace satisfying the* nth *Cayley–Hamilton identity.*

There is a formal property of trace which is used frequently and which we need to recall: If a, $b \in T$ are polynomials independent of a variable x (we think of the x_i's as variables) and $\operatorname{tr}(ax) = \operatorname{tr}(bx)$ then $a = b$.

2.8. We come now to the proof of the main theorem. This is in a way an anticlimax to the discussion since it is really a corollary of the previous theorems. Nevertheless it seems to have gone unnoticed until now.

Proof of Theorem 2.6. We present R as T/I where T is a free algebra in the category introduced in Theorem 3 of 2.7. We have the embedding $T \subseteq B = M_n(A)$ with $B^G = T$. We assume T has infinitely many variables for a technical reason (this, of course, is no restriction). Consider the two-sided ideal BIB of B. Since $B = M_n(A)$ we have $BIB = M_n(J)$ and J is a G invariant ideal of A. The induced map $R = T/I \to B/BIB = M_n(A/J)$ is the universal map j of R in $n \times n$ matrices and since G is linearly reductive j is surjective to $M_n(A/J)^G$. So we need only show that j is injective; that is to say, $BIB \cap T = I$. The proof of this fact will use the Reynolds' operator and will generalize the usual setting of invariant theory in the commutative case.

The Reynolds' operator r is the canonical G equivariant projection of B to B^G. This is really functorially defined on the category of rational G modules and it is natural in the sense that given two G modules M, N and a G map $f\colon M \to N$ we have a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r} \\ M^G & \xrightarrow{f|M^G} & N^G. \end{array}$$

In particular for the algebra B we have the following identities:

(1) If $a \in B^G$, $b \in B$ then $r(ab) = ar(b)$, $r(ba) = r(b)\,a$

(2) $r(\operatorname{tr}(a)) = \operatorname{tr}(r(a))$.

Furthermore we think of B as a multigraded algebra by the degrees in the variables x_i's and r preserves the multidegree (in fact more generally r commutes with the group of linear transformations in the vriables x_i's). So now assume $a = \sum a_i u_i b_i \in BIB \cap T$, $u_i \in I$, $a_i, b_i \in B$. We choose a variable x which does not appear in the elements a_i, u_i, b_i. We have $\operatorname{tr}(ax) = \operatorname{tr}(\sum a_i u_i b_i x) = \operatorname{tr}(\sum_i b_i x a_i u_i)$. Applying r we obtain

$$\operatorname{tr}(ax) = \operatorname{tr}\left(\sum r(b_i x a_i)\, u_i\right).$$

Now for each i, $r(b_i x a_i)$ is an invariant linear in x so by the first fundamental theorem we have

$$r(b_i x a_i) = \sum_j s_{ij} x t_{ij} + \sum_k \operatorname{tr}(m_{ik} x)\, n_{ik}$$

with s_{ij}, t_{ij}, m_{ik}, $n_{ik} \in T$. Thus $\operatorname{tr}(ax) = \operatorname{tr}(\sum_{ij} t_{ij} u_{ij} s_{ij} x + \sum_{ik} \operatorname{tr}(n_{ik} u_i)\, m_{ik} x)$; hence

$$a = \sum t_{ij} u_i s_{ij} + \sum \operatorname{tr}(n_{ik} u_i)\, m_{ik}.$$

Since I is closed under the race operation it follows that $a \in I$. This completes the proof of the theorem.

We note a curious corollary.

COROLLARY. *If R is a Q-algebra in which $r^n = 0$ for all r and a fixed n then R can be embedded in $n \times n$ matrices over a commutative ring.*

Proof. We think of R as an algebra with trace setting $\operatorname{tr}(r) = 0$ identically. The equaon $r^n = 0$ is now the nth Cayley–Hamilton identity. We can thus apply our general theorem.

Actually we have never clarified completely whether we were using algebras with 1 or not. This actually is inessential but if we want to remain in the class of algebras with 1 we can add 1 to R, set $\operatorname{tr}(1) = n$ and easily verify that the new algebra with 1 still verifies the nth Cayley–Hamilton identity directly. One should compare this result with [4].

We would like to close with some remarks on invariant theory. If G is a linearly reductive group acting on an affine variety with coordinate ring A the general theory gives that the ring of invariants A^G is the coordinate ring of an affine variety where geometric points correspond to closed orbits. In fact if $V = \mathrm{Spec}(A)$, $W = \mathrm{Spec}(A^G)$, and $\pi\colon V \to W$ is the corresponding map the fiber $\pi^{-1}(P)$ of a geometric point contains a unique closed orbit.

In our setting $T = M_n(A)^G$ and A^G is the center of T. The ideals of $M_n(A)$ are all of the form $M_n(J)$, J an ideal of A and the G invariant ideals of $M_n(A)$ correspond to the G invariant ideals of A. We have seen that each trace stable ideal I of T extends to $B = B_n(A)$ in an ideal $M_n(J)$ which gives the universal map $T/I \hookrightarrow M_n(A/J)$. So A/J is the coordinate ring of the space of n dimensional representations of T/I. The correspondence $I \to J$ has a particularly nice form for maximal ideals. Let me assume that the base field is algebraically closed. Let $I \subset T$ be maximal (as an ideal stable under trace). Then it is easily seen that T/I is a semisimple algebra. Contracting I to the center A^G of T we have a maximal ideal, i.e., a geometric point P_I of the quotient variety.

Proposition. *$BIB = M_n(J)$ where J is the ideal of the closed orbit in $\pi^{-1}(P_I)$.*

Proof. Consider the universal map $T/I \hookrightarrow M_n(A/J)$. Since the points of A/J correspond to complex representations of T/I and there is only one conjugacy class of such representation (given the trace) we have that the variety defined by J is the closed orbit. To prove that J is the ideal defining the closed orbit we need to show that A/J is smooth. We take the infinitesimal criterion for smoothness: Let B be a ring, $N \subset B$ an ideal with $N^2 = 0$ and $\rho\colon A \to B/N$ a point we want to lift ρ to $\tilde{\rho}\colon A \to B$. Now the composition $T/I \to M_n(A) \to M_n(B/N)$ gives a representation of T/I; its image is a semisimple algebra S isomorphic to T/I (since the traces of the idempotents are fixed). We lift a set of generators of S to $M_n(B)$, obtaining an algebra S' such that $S'/S' \cap M_n(N) = S$. Clearly $S' \cap M_n(N)$ is nilpotent. Since we are in characteristic zero we know that there is a lifting of S to S', $u\colon S \to S'$. So consider the diagram

$$\begin{tikzcd} T/I = S \arrow[rr, "u"] \arrow[dr] & & M_n(B) \\ & M_n(A) \arrow[r] \arrow[ur, dashed] & M_n(B/N). \end{tikzcd}$$

By the universal properties the map u factors and this gives, again by the universal property, the required lifting.

We may express the previous proposition in a more attractive form.

PROPOSITION. *The map $I \to BIB$ establishes a bijection between the maximal trace stable ideals of T and the maximal G stable ideals of $M_n(A)$.*

3. Semisimple Algebras

We develop as an application a description of the ideal defining a semisimple conjugacy class in $gl(n, \mathbb{C})$.

Let x be an $n \times n$ matrix, it is a very difficult problem to determine a complete set of generators for the defining ideal of the closure of the conjugacy class of x (cr. [2, 7] for some conjectures).

Here we want to show that for x semisimple the "obvious" equations generate the ideal, it will be clear that the method applies more generally for the orbit of an m-tuple of matrices $(x_1, ..., x_m)$ generating a semisimple representation and will be stated in this form.

Let A be the algebra of matrices (with 1), generated by $x_1, ..., x_m$. A is a semisimple algebra, by hypothesis, with trace induced by $n \times n$ matrices. By the previous proposition the universal map of A to $n \times n$ matrices gives as commutative ring exactly the coordinate ring of the "reduced" orbit of $(x_1, ..., x_m)$. Now this universal commutative ring can be explicitly presented: we choose generic matrices $\xi_i = (x_{hk}^{(i)})$ and substituting the ξ_i's with the x_i's we impose all the linear relations expressing A as a finite dimensional algebra with trace. The corresponding matrix equations give rise to a system of polynomial equations which define the prime ideal of the orbit. For instance for one semisimple matrix $A = \mathbb{C}[x]/(f(t))$, $f(t)$ is the minimal polynomial of x and the trace is given once we give the elements $a_k = \mathrm{Tr}(x^k)$, or equivalently the coefficients of the characteristic polynomial. Thus the equations are $\mathrm{Tr}(x^k) = a_k$, $1 \leqslant k < \deg f(x)$ and the entries of $f(x)$.

These polynomials generate the prime ideal of the corresponding conjugacy class.

4. Formal Smoothness

Let us consider the category $\mathscr{C}_n$ of algebras with trace satisfying the nth Cayley–Hamilton identity. Following Artin–Schelter [1] set.

DEFINITION. An algebra R in $\mathscr{C}_n$ is formally smooth if, given any algebra S in $\mathscr{C}_n$, a nilpotent ideal N in S and a map $f\colon R \to S/N$, f can be lifted to $f'\colon R \to S$.

In the previous 2 sections we have used this property for semisimple algebras!

THEOREM. *R is formally smooth if and only if, for the universal map $j: R \to M_n(A_R)$ we have A_R formally smooth (as a commutative algebra).*

Proof. First, assume R formally smooth. To prove this for A_R given a map $g: A_R \to C/N$, C commutative N nilpotent we must lift g.

Consider

$$j: \quad R \to M_n(A_R) \to M_n(C/N).$$

Since R is smooth we can lift it

$$\begin{tikzcd} R \arrow[r] \arrow[d, "j"'] & M_n(C) \arrow[d] \\ M_n(A_R) \arrow[r] & M_n(C/N) \end{tikzcd}$$

but then by the universal property of j we can lift g.

Conversely assume A_R smooth and let us give a diagram

$$\begin{tikzcd} & S \arrow[d] \\ R \arrow[r] & S/N \end{tikzcd}$$

N nilpotent, consider the universal maps

$$\begin{tikzcd} S \arrow[r] \arrow[d] & M_n(C) \arrow[d] \\ S/N \arrow[r] & M_n(C/J), \end{tikzcd}$$

$$R \longrightarrow M_n(A_R).$$

We cannot apply directly the hypothesis on A_R since J is usually not nilpotent. We need thus a further lemma on the Reynolds's operator which will show us that "S embeds in $M_n(C/J^k)$ for k large enough."

Once we do this we can lift the induced map $A_R \to C/J$ to $A_R \to C/J^k$ and we have a map $R \to M_n(C/J^k)$ which maps to the invariants of $M_n(C/J^k)$ under the group $GL(n)$. But S embeds into $M_n(C/J^k)$ and by the surjectivity of invariants under surjective maps we must have that S is the ring of invariants also in $M_n(C/J^k)$ completing the lifting argument.

Thus we have to prove the previous statement. We start with the trace ring T and a trace stable ideal I, let B be the universal matrix ring for T

and $J = BIB$ an ideal in B. We know that $J \cap T = I$ and now we look at $J^k \cap T$. This is not necessarily I^k but we claim that:

LEMMA. $J^{kn^2} \cap T \subseteq I^k$.

Proof. First, we make a formal remark. Suppose we have two trace rings T_1, T_2 relative to generic matrices (x_i), (y_j). Suppose we give a map $\psi: T_2 \to T_1$ sending the generic matrices y_j to elements u_j of T_1. This induces a G equivariant map of the corresponding universal rings B_1, B_2 and a commutative diagram of Reynold's operators

$$\begin{array}{ccc} B_2 & \longrightarrow & B_1 \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r} \\ T_2 & \longrightarrow & T_1. \end{array}$$

In particular let us add some generic matrices $y_1, y_2, \ldots, y_h$ to T_1 and let T_2 be the resulting ring. Then if $a_1, a_2, \ldots, a_{h+1} \in B_1$ and $u_1, \ldots, u_h \in T_1$ we can compute

$$r(a_1 u_1 a_2 u_2 \cdots a_h u_h a_{h+1}) \text{ by}$$

first computing $r(a_1\, y_1 a_2\, y_2 \cdots a_h\, y_h a_{h+1})$ and then substituting y_i with u_i.

Now r preserves the degrees in the generic matrices so that $r(a_1\, y_1 a_2\, y_2 \cdots y_h a_{h+1})$ is linear in each y_i and a sum of invariants. We know the general form of invariants and we can say that such a value is a sun

$$\operatorname{tr}(M_1) \operatorname{tr}(M_2) \cdots \operatorname{tr}(M_t)\, M_{t+1},$$

where each M_i is Q monomial of degree $\leqslant n^2$. Looking how the y_i's will distribute we see that each M_i can contain at most n^2 of the y_i's so that we get that M_{t+1} has at least $(k-t)\, n^2$ of the y_i's.

Now, if we substitute for the y_i's elements u_i in an ideal I closed under trace we see that $r(a_1 u_1 \cdots a_{h+1})$ lies in $I^{t+(k-t)n^2}$ and the power we get is at least I^k, if $h = kn^2$.

Now, we can complete the argument. $S = T/I$ and $S/N = T/L$ we know that $L^h \subset I$. We consider the universal map $S \to M_n(C)$ and $M_n(J) = M_n(C)\, JM_n(C)$. The formal argument on the Reynold's operator shows that

$$M_n(J)^{h \cdot n^2} \cap S \subset N^h = 0$$

so that the S embed in $M_n(C/J^{h \cdot n^2})$.

5. Remarks on the Cayley–Hamilton Identity

Let

$$P_n(x_1,\ldots,x_n)=\sum_{\sigma\in S_{n+1}}\varepsilon_\sigma\phi_\sigma(x_1,\ldots,x_n),$$

where we set

$$\phi_0(x_1,\ldots,x_n)=\mathrm{Tr}(x_{i_1}\cdots x_{i_k})\,\mathrm{Tr}(x_{j_1}\cdots x_{j_h})\cdots\mathrm{Tr}(x_{s_1}\cdots x_{s_p})\,x_{v_1}\cdots x_{v_m}$$

with

$$\sigma=(i_1\cdots i_k)(j_1\cdots j_n)\cdots(s_1\cdots s_p)(v_1\cdots v_m n+1).$$

This is the fully polarized form of the Cayley–Hamilton identity of degree n. We consider this as an element of the free algebra and set $t=\mathrm{Tr}(1)$ (an indeterminate) in the free algebra with trace.

Proposition.

$$P_n(x_1,\ldots,x_k,1,1,\ldots,1)=P_k(x_1,\ldots,x_k)\cdot\phi_{n,k}(t)$$

and

$$\phi_{n,k}(t)=(n-k)!\,(t-n)(t-n+1)\cdots(t-k-1).$$

Proof. First, let us remark that $P_n(x_1,\ldots,x_k,1,1,\ldots,1)$ is homogeneous of degree k in $x_1,\ldots,x_k$ and each monomial M in $x_1,\ldots,x_k$ is multiplied by a polynomial in t which has degree $n-k$ and leading coefficient $\pm(n-k)!$.

In fact the monomials that upon substituting $x_i=1$, $i=k+1,\ldots,n$, give the highest contribution to the t power as coefficient of M are exactly $\mathrm{Tr}(x_{i_1})\,\mathrm{Tr}(x_{i_2})\cdots\mathrm{Tr}(x_{i_{(n-k)}})\cdot M$, where $i_1,\ldots,i_{n-k}$ is a permutation of $k+1,\ldots,n$. Now if we evaluate this identity in $h\times h$ matrices $k+1\leqslant h\leqslant n$ we do get the substitution $t\to h$ and an identity of $h\times h$ matrices of degree $k<h$.

Since there are no such identities we must have that the resulting identity is formally zero which can happen only if h is a root of each of the coefficients of the monomials M. But since each such coefficient has degree $n-k$ and we have found $n, n-1,\ldots,k+1$ as roots it must be of the form $\pm(n-k)!\,(t-n)(t-n+1)\cdots(t-k-1)$.

Now $P_n(x_1,\ldots,x_k,1,\ldots,1)=\phi(t)\cdot Q(x_1,\ldots,x_k)$ where Q is not formally zero and actually starts with the monomial $\sum x_{\sigma(1)}\cdots x_{\sigma(k)}+$ trace terms.

Since evaluating P_n in $k\times k$ matrices with $t=k$ we get $\phi(k)\neq 0$ we deduce that Q is a trace identity but by the uniqueness of trace identities of $k\times k$ matrices in degree k we deduce $Q=P_k$.

COROLLARY 1. $P_n(1, 1, \ldots, 1) = \pm n!(t-n)(t-n+1)\cdots(t-1)$.

COROLLARY 2. *If R is a trace algebra satisfying the nth Cayley–Hamilton identity and* $\operatorname{Tr}(1) = k < n$ *then R satisfies the kth Cayley–Hamilton identity.*

Proof. *R satisfies $P_n(x_1, \ldots, x_k, 1, \ldots, 1)$* which is a nonzero multiple of P_k.

COROLLARY 3. *If R is a trace algebra with* 1 *satisfying the nth Cayley–Hamilton identity then R can be canonically decomposed as $R = \oplus_{i=1}^{n} R_i$ where each summands R_i has the property that, calling l_i its unit element* $\operatorname{Tr}(l_i) = i \cdot l_i$. *(We will drop the summands $R_i = 0$.)*

Proof. We have that $t_0 = \operatorname{tr}(1)$ satisfies the identity $(t-n)(t-n+1)\cdots(t-1)$ so the ring $Q(t_0)$ is a direct sum of fields in each one of which $\operatorname{Tr}(l_i) = i \cdot l_i$ (some may not appear). This produces the required decomposition of R.

SUMMARIZING

A ring R satisfying the nth Cayley–Hamilton identity decomposes canonically as $R_1 \oplus R_2 \oplus \cdots \oplus R_n$ (where some R_i might be 0). Each R_i satisfies the ith Cayley–Hamilton identity and its unit l_i has trace i.

In particular for the free algebra we have.

The free algebra in the category $\mathscr{C}_n$ is the direct sum $T_1 \oplus T_2 \oplus \cdots \oplus T_n$ of the trace rings of generic matrices (in the appropriate set of variables).

This actually corrects an incorrect statement in my papers [4, 5]. But if we work with algebras without 1 then the free algebra is just the augmentation ideal of T_n as used in these papers.

REFERENCES

1. M. ARTIN AND W. SCHELTER, Smooth algebras (preprint).
2. C. DE CONCINI AND C. PROCESI, Symmetric functions, conjugacy classes and the flag variety. *Invent. Math.* **64** (1981), 203–219.
3. C. PROCESI, "Rings with Polynomial Identities," Lecture Notes in Pure and Appl. Math., Vol. 17, Dekker, New York, 1979.
4. C. PROCESI, The invariant theory of $n \times n$ matrices, *Advan. in Math.* **19** (1976), 306–381.
5. C. PROCESI, Trace identities and standard diagrams, "Ring Theory," Lecture Notes in Pure and Applied Math., Vol. 51, Dekker, New York 1979.
6. L. SMALL, An example in *PI*-rings, *J. Algebra* **17** (1971), 434–436.
7. T. TANISAKI, Defining ideals of the closures of the conjugacy classes and representations of the Weyl groups, *Tôhoku Math. J.* **34** (1982), 575–585.